

# Note on sampling without replacing from a finite collection of matrices

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This technical note supplies an affirmative answer to a question raised in a recent pre-print in the context of a “matrix recovery” problem. Assume one samples  $m$  Hermitian matrices  $X_1, \dots, X_m$  with replacement from a finite collection. The deviation of the sum  $X_1 + \dots + X_m$  from its expected value in terms of the operator norm can be estimated by an “operator Chernoff-bound” due to Ahlswede and Winter. The question arose whether the bounds obtained this way continue to hold if the matrices are sampled without replacement. We remark that a positive answer is implied by a classical argument by Hoeffding. Some consequences for the matrix recovery problem are sketched.

This is a technical comment on [1]. While we provide a (minimal) introduction, readers not familiar with [1] may find the present note hard to follow.

## A. Motivation

The *low-rank matrix recovery problem* [1–10] is: Reconstruct a low-rank matrix  $\rho$  from  $m$  randomly selected matrix elements. The more general version introduced in [1] reads: Reconstruct  $\rho$  from  $m$  randomly selected expansion coefficients with respect to any fixed matrix basis.

Let us consider what seems to be the most mundane aspect of the problem: the way in which the  $m$  coefficients are “randomly selected”. Assume we are dealing with an  $n \times n$  matrix  $\rho$ . The statement of the matrix recovery problem calls for us to sample  $m$  of the  $n^2$  coefficients characterizing  $\rho$  *without replacing*. This yields a random subset  $\Omega$  consisting of  $m$  of the  $n^2$  coefficients, from which the matrix  $\rho$  is then to be recovered.

Due to the requirement that the drawn coefficients be distinct, the  $m$  samples are not independent. Their dependency turns out to impede the technical analysis of the recovery algorithms. In order to avoid this complication, most authors chose to first analyze a variant where the revealed coefficients are drawn independently and then, in a second step, relate the modified question to the original one. Two such proxies for sampling without replacement have been discussed:

1. *The Bernoulli model* [3, 4, 10]. Here, each of the  $n^2$  coefficients is assumed to be known with probability  $\frac{m}{n^2}$ . Thus the number of revealed coefficients is itself a random variable (with expectation value  $m$ ). The minor draw-back of this approach is that, with finite probability, significantly more than  $m$  coefficients will be uncovered. These possible violations of the rules of the original problem have to be factored in, when the success probability of the algorithm is computed.

2. *The i.i.d. approach* [1, 8, 9]. The known coefficients are obtained by sampling  $m$  times *with* replacement. The draw-back here is that, with fairly high probability, some coefficients will be selected more than once. To understand why

this is undesirable, we need to recall some technical definitions from [1].

Let  $A_1, \dots, A_m$  be random variables taking values in  $[1, n^2]$ . For now, assume the  $A_i$ ’s are distributed uniformly and independently. Let  $\{w_a\}_{a=1}^{n^2}$  be an orthonormal Hermitian basis in the space of  $n \times n$ -matrices. A central object in the analysis is the *sampling operator*, defined as

$$\mathcal{R} : \rho \mapsto \frac{n^2}{m} \sum_{i=1}^m \text{tr}(\rho w_{A_i}) w_{A_i}. \quad (1)$$

If the  $A_i$  are all distinct, then  $\frac{n^2}{m} \mathcal{R}$  is a projection operator. If, on the other hand, some basis elements occur more than once, the spectrum of the sampling operator will be more complicated. More importantly, the operator norm  $\|\frac{n^2}{m} \mathcal{R}\|$  may become fairly large. The latter effect is undesirable, as the logarithm of the operator norm appears as a multiplicative constant in the final bound on the number of coefficients which need to be known in order for the reconstruction process to be successful.

There seem to be three ways to cope with this problem. First, use the worst-case estimate  $\|\frac{n^2}{m} \mathcal{R}\| \leq m$  (done in Section II.C of [1]). Second, use the fact that the operator norm is very likely to be of order  $O(\log n)$  (suggested at the end of Section II.C in [1] and implemented in later versions of [9]). Third, prove that the arguments in [1] remain valid when the  $A_i$ ’s are chosen without replacement. Supplying such a proof is the purpose of the present note.

Following earlier work [3, 4], Ref. [1] reduces the analysis of the matrix recovery problem to the problem of controlling the operator norm of various linear functions of  $\mathcal{R}$  (c.f. Lemma 4 and Lemma 6 of [1]). This, in turn, is done by employing a large-deviation bound for the sum of independent matrix-valued random variables, which was derived in [11]. Below, we point out that in some situations this bound remains valid when the random variables are not independent, but represent sampling without replacing.

## B. Statement

Let  $C$  be a finite set. For  $1 \leq m \leq |C|$ , let  $X_i$  be a random variable taking values in  $C$  with uniform probability. We assume that all the  $X_i$  are independent, so that  $\mathbf{X} = \langle X_1, \dots, X_m \rangle$  is a  $C^m$ -valued random vector modeling sampling *with* replacement from  $C$ . Likewise, let  $\mathbf{Y} =$

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$\langle Y_1, \dots, Y_m \rangle$  be a random vector of  $C$ 's sampled uniformly without replacement.

We are mainly interested in the case where  $C$  is a finite set of Hermitian matrices with some additional properties: We assume the set is centered  $\mathbb{E}[X_i] = 0$  and that there are constants  $c, \sigma_0 \in \mathbb{R}$  bounding the operator norm  $\|X_i\| \leq c$  and the variance  $\|\mathbb{E}[X_i^2]\| \leq \sigma_0^2$  of the random variables. Then:

**Theorem 1** (Operator-Bernstein inequality). *With the definitions above, let  $S_{\mathbf{X}} = \sum_{i=1}^m X_i$  and  $S_{\mathbf{Y}} = \sum_{i=1}^m Y_i$ . Let  $V = m\sigma_0^2$ . Then for both  $S = S_{\mathbf{X}}$  and  $S = S_{\mathbf{Y}}$  it holds that*

$$\Pr[\|S\| > t] \leq 2n \exp\left(-\frac{t^2}{4V}\right), \quad (2)$$

for  $t \leq 2V/c$ , and

$$\Pr[\|S\| > t] \leq 2n \exp\left(-\frac{t}{2c}\right), \quad (3)$$

for larger values of  $t$ .

The version involving  $S_{\mathbf{X}}$  has been proved in [1] as a minor variation of the operator-Chernoff bound from [11]. In the proof, the failure probability is bounded from above in terms of the “operator moment-generating function”

$$M_{\mathbf{X}}(\lambda) = \mathbb{E}[\text{tr} \exp(\lambda S_{\mathbf{X}})].$$

To establish the more general statement, it would be sufficient to show that  $M_{\mathbf{Y}} \leq M_{\mathbf{X}}$ . In fact, this relation is well-known to hold for real-valued random variables. One popular way of proving it involves the notion of *negative association* [12, 13]. Indeed, the author of [1] tried to generalize this concept to the case of matrix-valued random variables, but failed to overcome its apparent dependency on the *total* order of the real numbers. However, he overlooked a much older and more elementary argument given in [14], which only relies on certain convexity properties and applies without change to the matrix-valued case (see below).

### C. Implications

As a consequence of Theorem 1, the analysis in Section II.C of [1] can be simplified and improved, by setting the constant  $C$  equal to one. The remark at the end of that section applies. In particular, in the rest of that paper, one may assume that  $\|\Delta_T\|_2 < n^{1/2} \|\Delta_T^\perp\|_2$ . Thus, the conditions on the certificate  $Y$  in Section II.E may be relaxed to  $\|\mathcal{P}_T Y - \text{sgn } \rho\|_2 \leq \frac{1}{2n^{1/2}}$ . This implies that  $l$ , the number of iterations of the “golfing scheme”, may be reduced to  $l = \lceil \log_2(2n^{1/2} \sqrt{r}) \rceil$ . The estimates on  $|\Omega|$  in Theorems 1, 2, and 3 therefore all improve by a factor of  $\frac{\log_2 n^2}{\log_2 n^{1/2}} = 4$ .

In [9], Proposition 3.3 becomes superfluous. The final bounds improve accordingly.

The consequences are more pronounced for an upcoming detailed analysis [15] of noise resilience (in the spirit of [16]) of quantum mechanical applications.

The present note makes no statements about approaches which either rely on the Bernoulli model, or use the non-commutative Kintchine inequality instead of the operator Chernoff bound [3, 4, 10].

Finally, note that the “golfing scheme” employed in [1, 8] demands that  $l$  independent batches of coefficients be sampled. As a consequence of Theorem 1, every single batch may be assumed to be drawn without replacement. However, for technical reasons, it is still necessary that the batches remain independent. This does not constitute a problem. Indeed, let  $\Omega$  be the set of distinct coefficients used by the golfing scheme. It is shown that, with high probability, there exists a “dual certificate” in the space spanned by the basis elements corresponding to the coefficients in  $\Omega$ . Since  $\Omega$  is just a random subset of cardinality  $|\Omega| \leq m$ , the probability that there is a dual certificate in the space spanned by  $m$  distinct random basis elements (obtained from sampling without replacing) can only be higher. A very similar argument has recently been given in [10], where the golfing scheme has been modified to work with the Bernoulli model.

### D. Proof

In this section, we repeat an argument from [14] which implies that for all  $\lambda \in \mathbb{R}$  the inequality  $M_{\mathbf{Y}}(\lambda) \leq M_{\mathbf{X}}(\lambda)$  holds. We emphasize that the proof of [14] does not need to be modified in order to apply matrix-valued random variables. However, the version given below makes some steps explicit which were omitted in the original paper.

For now, let  $C$  be any finite set; let  $\mathbf{X}, \mathbf{Y}$  be as above.

The central observation is that one can generate the distribution of  $\mathbf{X}$  by first sampling  $\mathbf{y} = \langle y_1, \dots, y_m \rangle$  without replacement, and then drawing the  $\langle x_1, \dots, x_m \rangle$  from  $\{y_1, \dots, y_m\}$  in a certain (unfortunately not completely trivial) way.

To make that second step precise, we introduce a random partial function  $\mathbf{Z}$  from  $C^m$  to  $C^m$ . The domain of  $f$  is the set of vectors  $\mathbf{y} \in C^m$  with pairwise different components ( $y_i \neq y_j$ ). Given such a vector  $\mathbf{y}$ , we sequentially assign values to the components  $Z_1, \dots, Z_m$  of  $\mathbf{Z}(\mathbf{y})$  by sampling from  $\{y_1, \dots, y_m\}$  according to the following recipe. At the  $k$ th step, let  $D_k$  be the subset of  $\{y_1, \dots, y_m\}$  of values which have already been drawn in a previous step. To get  $Z_k$ :

1. with probability  $\frac{|D_k|}{|C|}$  take a random element from  $D_k$ , and
2. with probability  $1 - \frac{|D_k|}{|C|}$  take a random element from the  $\{y_1, \dots, y_m\}$  not contained in  $D_k$ .

(Here, by a “random” element, we mean one sampled uniformly at random from the indicated set). Then

**Lemma 2.** *With the definitions above,  $\mathbf{X}$  and  $\mathbf{Z}(\mathbf{Y})$  are identically distributed.*

What is more, if  $C$  is a subset of a vector space, then

$$\mathbb{E}_{\mathbf{Z}} \left[ \sum_{i=1}^m Z_i(\mathbf{Y}) \right] = \sum_{i=1}^m Y_i. \quad (4)$$

*Proof.* Choose  $k \in \{1, \dots, m\}$ , let  $\mathbf{x} \in C^m$ . We compute the conditional probability

$$\Pr[Z_k(\mathbf{Y}) = x_k \mid Z_1(\mathbf{Y}) = x_1, \dots, Z_{k-1}(\mathbf{Y}) = x_{k-1}].$$

If there is a  $j < k$  such that  $x_k = x_j$ , then, according to the first rule above, the probability is

$$\frac{|D_k|}{|C|} \frac{1}{|D_k|} = \frac{1}{|C|}.$$

Otherwise, by the second rule, the probability reads

$$\left(1 - \frac{|D_k|}{|C|}\right) \frac{1}{|C| - |D_k|} = \frac{1}{|C|}$$

as well. Iterating:

$$\begin{aligned} & \Pr[Z_1(\mathbf{Y}) = x_1, \dots, Z_m(\mathbf{Y}) = x_m] \\ &= \Pr[Z_1(\mathbf{Y}) = x_1, \dots, Z_{m-1}(\mathbf{Y}) = x_{m-1}] \frac{1}{|C|} \\ &= \Pr[Z_1(\mathbf{Y}) = x_1, \dots, Z_{m-2}(\mathbf{Y}) = x_{m-2}] \frac{1}{|C|^2} \\ &= \dots = \frac{1}{|C|^m}. \end{aligned}$$

This proves the first claim.

We turn to the second statement. The left hand side of (4) is manifestly a linear combination of the random variables  $Y_i$ . From the definition of  $\mathbf{Z}$ , it is also invariant under any permutation  $Y_i \mapsto Y_{\pi(i)}$ . As a linear and symmetric function, it is of

the form  $K \sum_i^m Y_i$  for some constant  $K$ . To compute  $K$ , we use the fact that the  $Y_i$  are identically distributed, so that

$$\mathbb{E}_{\mathbf{Y}} \left[ \mathbb{E}_{\mathbf{Z}} \left[ \sum_{i=1}^m Z_i(\mathbf{Y}) \right] \right] = m \mathbb{E}[Y_1],$$

$$\mathbb{E}_{\mathbf{Y}} \left[ K \sum_{i=1}^m Y_i \right] = Km \mathbb{E}[Y_1].$$

Thus  $K = 1$  and we are done.  $\square$

Now let  $f$  be a convex function on the convex hull of  $C$ . Using Jensen's inequality and Lemma 2,

$$\begin{aligned} \mathbb{E}_{\mathbf{X}} \left[ f \left( \sum_{i=1}^m X_i \right) \right] &= \mathbb{E}_{\mathbf{Y}} \mathbb{E}_{\mathbf{Z}} \left[ f \left( \sum_{i=1}^m Z_i(\mathbf{Y}) \right) \right] \\ &\geq \mathbb{E}_{\mathbf{Y}} \left[ f \left( \mathbb{E}_{\mathbf{Z}} \left[ \sum_{i=1}^m Z_i(\mathbf{Y}) \right] \right) \right] \\ &= \mathbb{E}_{\mathbf{Y}} \left[ f \left( \sum_{i=1}^m Y_i \right) \right]. \end{aligned}$$

Finally, specialize to the case where  $C$  is a finite set of Hermitian matrices. Since the function  $c \mapsto \text{tr} \exp(\lambda c)$  is convex on the set of Hermitian matrices for all  $\lambda \in \mathbb{R}$ , any upper bound on moment generating functions derived for matrix-valued sampling with replacing is also valid for sampling without replacing.

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